5.5: Substitution Method (Indefinite Integrals)

Recall: Chain Rule

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\cdot g'(x). \quad \text{or} \quad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}.$$

We ~~use~~ wish to run the chain rule backwards.

Theorem: (The Substitution Rule)

$$\int f(g(x))\cdot g'(x)\,dx = \int f(u)\,du. \quad (\text{where } u = g(x)).$$

Ex(1): $\int (x^3+x)^5\cdot(3x^2+1)\,dx.$

Let $u=(x^3+x)$. Then $du = 3x^2+1\,dx$.

Thus
$$\int (x^3+x)^5(3x^2+1)\,dx = \int u^5\cdot du = \frac{u^6}{6}+C = \frac{(x^3+x)^6}{6}+C$$

Ex(2): $\int \sqrt{2x+1}\,dx$

Let $u = 2x+1$. Then $du = 2dx$ or $dx = \frac{du}{2}$.

Thus
$$\int\sqrt{2x+1}\,dx = \int\sqrt{u}\cdot\frac{du}{2} = \frac{1}{2}\int\sqrt{u}\cdot du = \frac{1}{2}\,\frac{u^{3/2}}{3/2}+C = \frac{1}{3}(2x+1)^{3/2}+C.$$

Ex(3): $\int \sec^2(5x+1)\cdot 5\,dx$ *

Let $u = 5x+1$. Then $du = 5dx$

Thus
$$\int\sec^2(5x+1)\cdot 5\,dx = \int\sec^2 u\cdot du = \tan u + C = \tan(5x+1)+C$$

Ex(4): $\int x^2 e^{x^3} dx$.

Let $u = x^3$. Then $du = 3x^2 dx$ or $x^2 dx = \frac{du}{3}$.

Thus $\int x^2 e^{x^3} dx = \int e^u \frac{du}{3} = \frac{1}{3} e^u + C = \frac{1}{3} e^{x^3} + C$.

Ex(5): Using Trig Identities.

(a) $\int \sin^2 x \, dx \underset{\text{Trig Iden.}}{=} \int \frac{1-\cos 2x}{2} dx = \int \frac{1}{2} dx - \int \frac{\cos 2x}{2} dx$

$= \frac{1}{2} x - \int \frac{\cos 2x}{2} dx$.

Let $u = 2x$ Then $du = 2dx \Rightarrow dx = \frac{du}{2}$.

Thus $\int \frac{\cos 2x}{2} dx = \int \frac{\cos u}{2} \cdot \frac{du}{2} = \frac{\sin u}{4} + C = \frac{\sin 2x}{4} + C$

Therefore $\int \sin^2 x \, dx = \frac{1}{2} x - \frac{\sin 2x}{4} + C$.

(b) Similarly $\int \cos^2 x \, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$ (Use $\cos^2 x = \frac{1+\cos 2x}{2}$)

(c) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$. Let $u = \cos x$ then $du = -\sin x \, dx$.

So $\int \frac{\sin x}{\cos x} dx = \int -\frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C$.

Ex(6): Trying Different Methods

$\int \frac{2z \, dz}{\sqrt[3]{z^2+1}}$

Method 1: $u = z^2+1$

$du = 2z \, dz$

$\int \frac{du}{\sqrt[3]{u}} = \frac{u^{2/3}}{2/3} + C = \frac{3}{2}(z^2+1)^{2/3} + C$

Method 2: $u = \sqrt[3]{z^2+1}$

$du = \frac{1}{3}(z^2+1)^{-1/3} \cdot 2z \, dz$

$\int \frac{3u^2 \, du}{u} = 3\int u \, du = 3\frac{u^2}{2} + C$

$= \frac{3}{2}\left(\sqrt[3]{z^2+1}\right)^2 + C$

$= \frac{3}{2}(z^2+1)^{2/3} + C$